Preliminary

Track Parameter Error Matrix Propagation in Matter and Magnetic Field, Error Matrices Combination

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Abstract

The goal of the following is to expose how we match local muon tracks with central tracker tracks. We first develop the infinitesimal position and momentum propagation of a particle in magnetic field and in matter. Then we derive the error matrix propagation in the 6 following variables: (position,momentum) = (x, y, z, px, py, pz) in cartesian coordinates. Those calculations are performed in the matrix formalism. In the third section we use this error matrix propagation to combine local muon error matrix computed in the muon system with the central tracker error matrix.

In the following, $\vec{x}^{\,in}$ and $\vec{p}^{\,in}$ ($\vec{x}^{\,out}$ and $\vec{p}^{\,out}$) represent the particle incoming (outcoming) position and momentum. All these quantities are considered as random variables with a mean value and an error.

1 Position and Momentum After an Infinitesimal Step:

1.1 Propagation in Magnetic Field:

We consider the propagation of a particle along its momentum direction performing an infinitesimal step dx in a region with magnetic field $\vec{B}$ (see Figure 1).

- q and m are the charge and mass of the considered particle travelling at speed v.
- We define the unit vector $\vec{u} = \frac{\vec{p}^{\,in}}{||\vec{p}^{\,in}||}$.

According to these assumptions, we can write:

$$\frac{d\vec{p}}{dt} = \frac{q}{\gamma m}\vec{p} \times \vec{B} \text{ and } dt = \frac{dx}{v}$$

and then, flipping in the cross product $\vec{B}$ and $\vec{p}$:

$$\delta\vec{p} = \delta\vec{\Omega}^B \times \vec{p} \text{ with } \delta\vec{\Omega}^B = -\frac{q}{\gamma m v}\vec{B}\delta x.$$

At first order:

$$\vec{p}^{\,out} = \vec{p}^{\,in} + \delta\vec{\Omega}^B \times \vec{p}^{\,in} \;,\; \delta\vec{\Omega}^B = -0.3\frac{q}{||\vec{p}^{\,in}||}\vec{B}\delta x$$

with δx in meter, B in Tesla, p in GeV and q = ± 1.

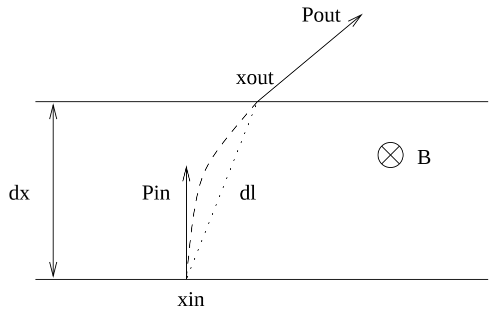

Figure 1: small step propagation in magnetic field

Within the approximation of a circular trajectory, one can easily calculate the outcoming position $\vec{x}^{\,out}$ as a function of the incoming $\vec{x}^{\,in}$:

$$\vec{x}^{\,out} = \vec{x}^{\,in} + \vec{dl} \simeq \vec{x}^{\,in} + \frac{dx}{2||\vec{p}^{\,in}||}(\vec{p}^{\,in} + \vec{p}^{\,out})$$

1.2 Propagation in Matter with Multiple Scattering and Energy Loss:

Let us consider a plane of matter with multiple Coulomb scattering (see Figure 2).

- $\vec{z}$ is a given particular direction not parallel to the momentum (for example: beam axis in the barrel part or y axis in the endcap part of the detector).
- We define $\vec{v}^o = \frac{\vec{z}\times\vec{u}}{||\vec{z}\times\vec{u}||}$ and $\vec{w}^o = \vec{v}^o \times \vec{u}$ with $\vec{u} = \frac{\vec{p}^{\,in}}{||\vec{p}^{\,in}||}$.

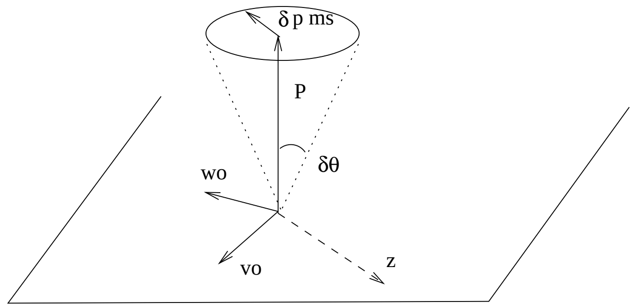

Figure 2: effect of multiple scattering on momentum direction

The multiple scattering circle around $\vec{p}$ is now in the $(\vec{v}^o, \vec{w}^o)$ plane as $\delta\vec{p}^{MS}$:

$$
\begin{aligned}
\delta\vec{p}^{MS} &= \delta\theta||\vec{p}^{\,in}||(cos\varphi\vec{v}^{o} + sin\varphi\vec{w}^{o}) \\
&= \delta\theta||\vec{p}^{\,in}||(\frac{cos\varphi}{||\vec{z}\times\vec{p}^{\,in}||}\vec{z} + \frac{sin\varphi}{||\vec{p}^{\,in}||}\vec{v}^{o}) \times \vec{p}^{\,in}
\end{aligned}
$$

where $\delta\theta$ and φ are two random quantities characterizing the multiple scattering process. Defining a new vector $\vec{v} = \vec{z} \times \vec{u}$, one can express $\vec{v}^{o}$ as $\vec{v}^{o} = \frac{\vec{v}}{||\vec{z}\times\vec{p}^{\,in}||}||\vec{p}^{\,in}||$ and get:

$$
\delta\vec{p}^{MS} = \frac{\delta\theta||\vec{p}^{\,in}||}{||\vec{z}\times\vec{p}^{\,in}||}(cos\varphi\vec{z} + sin\varphi\vec{v}) \times \vec{p}^{\,in}
$$

So one writes the multiple scattering effect as:

$$
\vec{p}^{\,out} = \vec{p}^{\,in} + \delta\vec{\Omega}^{MS} \times \vec{p}^{\,in} \text{ with } \delta\vec{\Omega}^{MS} = \frac{\delta\theta}{||\vec{v}||}(cos\varphi\vec{z} + sin\varphi\vec{v}).
$$

One now focuses on the energy loss process.

- Energy loss in matter of length of L is characterized by a mean value $\bar{E}_{loss}$ and a variance $\sigma_{E_{loss}}$.
- One can define the random energy loss e_{loss} in dx as the variable with mean value $\bar{e}_{loss} = \frac{dx}{L}\bar{E}_{loss}$ and variance: $\sigma_{e_{loss}} = \sqrt{\frac{dx}{L}}\sigma_{E_{loss}}$ since energy loss is a stochastic process.

In principle, $\bar{e}_{loss}$ and $\sigma_{e_{loss}}$ are momentum dependent. To give an order of magnitude of the energy loss in a calorimeter, with $L = 200cm$, $\bar{E}_{loss} = 3GeV$ and $\sigma_{E_{loss}} = 0.4GeV$. Within this notation, one can write the energy loss effect on momentum as:

$$
\vec{p}^{\,out} = \vec{p}^{\,in}\left(1 + \frac{e_{loss}}{||\vec{p}^{\,in}||}\right)
$$

And finally:

$$
\vec{p}^{\,out} = \vec{p}^{\,in}\left(1 + \frac{e_{loss}}{||\vec{p}^{\,in}||}\right) + \delta\vec{\Omega}^{MS} \times \vec{p}^{\,in} \text{ with } \delta\vec{\Omega}^{MS} = \frac{\delta\theta}{||\vec{v}||}\left(cos\varphi\vec{z} + sin\varphi\vec{v}\right).
$$

1.3 Summary: Propagation in Matter and Magnetic Field:

$$
\begin{aligned}
\vec{p}^{\,out} &= \vec{p}^{\,in}\left(1 + \frac{e_{loss}}{||\vec{p}^{\,in}||}\right) + \delta\vec{\Omega}^{B} \times \vec{p}^{\,in} + \delta\vec{\Omega}^{MS} \times \vec{p}^{\,in} && (1) \\
\vec{x}^{\,out} &= \vec{x}^{\,in} + \frac{dx}{||\vec{p}^{\,in}||}\left(\vec{p}^{\,in} + \frac{\delta\vec{\Omega}^{B}}{2} \times \vec{p}^{\,in} + \frac{\delta\vec{\Omega}^{MS}}{2} \times \vec{p}^{\,in}\right) && (2) \\
\text{with } \delta\vec{\Omega}^{B} &= -0.3\frac{q}{||\vec{p}^{\,in}||}\vec{B} && (3) \\
\text{and } \delta\vec{\Omega}^{MS} &= \frac{\delta\theta}{||\vec{v}||}\left(cos\varphi\vec{z} + sin\varphi\vec{v}\right) && (4)
\end{aligned}
$$

where $\vec{u} = \frac{\vec{p}^{in}}{||\vec{p}^{in}||}$ and $\vec{v} = \vec{z} \times \vec{u}$. Transforming the cross product into matrix product (see Appendix), we get the following antisymmetric matrices $\delta\Omega^B$ and $\delta\Omega^{MS}$:

$$\begin{aligned} \delta\Omega^B &= -0.3\frac{q}{||\vec{p}^{in}||}B\ dx \\ \delta\Omega^{MS} &= \frac{\delta\theta}{||\vec{z} \times \vec{u}||}(cos\varphi Z + sin\varphi V) \end{aligned}$$

where B, Z and V are the antisymmetric matrices associated to the vectors $\vec{B}$, $\vec{z}$, and $\vec{v}$. The equations 1 and 2 lead to the following formula calculating the mean values $\bar{x}^{\,out}$ and $\bar{p}^{\,out}$ of $\vec{x}^{\,out}$ and $\vec{p}^{out}$ at first order:

$$\begin{aligned} \bar{x}^{\,out} &= \bar{x}^{\,in} + \frac{dx}{||\vec{p}^{in}||}\left(\bar{p}^{\,in} + \frac{\delta\Omega^B}{2}\bar{p}^{\,in}\right) \\ \bar{p}^{\,out} &= \bar{p}^{\,in}\left(1 + \frac{dx}{L}\frac{\bar{E}_{loss}}{||\vec{p}^{in}||}\right) + \delta\Omega^B\bar{p}^{\,in} \end{aligned}$$

since $< \delta\Omega^{MS} >= 0$ because $< cos\varphi >= 0$ and $< sin\varphi >= 0$.

2 Error Matrix Propagation:

The error matrix is defined as $E^{PP}_{ij} =< (p_i - \bar{p}_{\,i})(p_j - \bar{p}_{\,j}) >=< \delta p\ {}^t(\delta p) >$ where the δp's are the errors in p. The total matrix is a 6×6 symmetric matrix in the (momentum P, position X) variables:

$$E_{6\times6} = \begin{pmatrix} E^{PP}_{3\times3} & E^{PX}_{3\times3} \\ & \\ E^{XP}_{3\times3} & E^{XX}_{3\times3} \end{pmatrix} \text{ with } E^{XP} = {}^tE^{PX}$$

2.1 Propagation of E^{PP}:

We first concentrate on the propagation of momentum variables. Here the *in* index refers to the incoming quantities and the *out* index to the outcoming ones. Because $\delta\Omega^B$ and $\delta\Omega^{MS}$ are antisymmetric, ${}^t(\delta\Omega^B) = -\delta\Omega^B$ and ${}^t(\delta\Omega^{MS}) = -\delta\Omega^{MS}$. We define:

$$\delta e_{loss} = e_{loss} - \bar{e}_{loss}.$$

Note that unlike $\delta\Omega^B$ and $\delta\Omega^{MS}$, δe_{loss} is a one dimensional variable and not a matrix, measured in GeV. Thus one has at first order:

$$\begin{aligned} E^{PPout} &= < \delta p^{out}\ {}^t(\delta p^{out}) > \\ \text{with } \delta p^{out} &= \delta p^{in} + \delta\Omega^B\delta p^{in} + \delta\Omega^{MS}p^{in} + \frac{\delta e_{loss}}{||\vec{p}^{in}||}p^{in} \\ \text{and } {}^t(\delta p^{out}) &= {}^t(\delta p^{in}) - {}^t(\delta p^{in})\delta\Omega^B - {}^t(p^{in})\delta\Omega^{MS} + \frac{\delta e_{loss}}{||\vec{p}^{in}||}{}^t(p^{in}) \end{aligned}$$

Noting that $<\delta\Omega^B>=\delta\Omega^B$ because the magnetic field is assumed to be constant in the considered domain, we get:

$$\begin{aligned}
E^{PPout} &= <\delta p^{in\ t}(\delta p^{in})> - <\delta p^{in\ t}(\delta p^{in})> \delta\Omega^B + \delta\Omega^B <\delta p^{in\ t}(\delta p^{in})> \\
&\quad - <\delta p^{in\ t}(p^{in})\delta\Omega^{MS}> + <\delta\Omega^{MS} p^{in\ t}(\delta p^{in})> - \delta\Omega^B <\delta p^{in\ t}(\delta p^{in})> \delta\Omega^B \\
&\quad - \delta\Omega^B <\delta p^{in\ t}(p^{in})\delta\Omega^{MS}> - <\delta\Omega^{MS} p^{in\ t}(\delta p^{in})> \delta\Omega^B - <\delta\Omega^{MS} p^{in\ t}(p^{in})\delta\Omega^{MS}> \\
&\quad + <\frac{\delta e_{loss}}{||\vec{p}^{in}||}\delta p^{in\ t}(p^{in})> + <\frac{\delta e_{loss}}{||\vec{p}^{in}||}\delta\Omega^{MS} p^{in\ t}(p^{in})> + <\frac{\delta e_{loss}}{||\vec{p}^{in}||}\delta\Omega^B \delta p^{in\ t}(p^{in})> \\
&\quad + <\frac{\delta e_{loss}}{||\vec{p}^{in}||} p^{in\ t}(\delta p^{in})> - <\frac{\delta e_{loss}}{||\vec{p}^{in}||} p^{in\ t}(\delta p^{in})> \delta\Omega^B \\
&\quad - <\frac{\delta e_{loss}}{||\vec{p}^{in}||} p^{in\ t}(p^{in})\delta\Omega^{MS}> + <\left(\frac{\delta e_{loss}}{||\vec{p}^{in}||}\right)^2 p^{in\ t}(p^{in})> .
\end{aligned}$$

Nevertheless $<\delta\Omega^{MS}>=0$ and $\delta\Omega^{MS}$ is independent on δp^{in}. The same remark is also true for δe_{loss} and $<\delta e_{loss}>=0$. Thus a lot of terms of the previous equation vanish. Writing $E^{PPin}=<\delta p^{in\ t}(\delta p^{in})>$, it reads:

$$\begin{aligned}
E^{PPout} &= E^{PPin} - E^{PPin}\delta\Omega^B + \delta\Omega^B E^{PPin} - \delta\Omega^B E^{PPin}\delta\Omega^B \\
&\quad - <\delta\Omega^{MS} p^{in\ t}(p^{in})\delta\Omega^{MS}> + <\left(\frac{\delta e_{loss}}{||\vec{p}^{in}||}\right)^2 p^{in\ t}(p^{in})>
\end{aligned}$$

where we define $E^{MS}=<\delta\Omega^{MS} p^{in\ t}(p^{in})\delta\Omega^{MS}>$. With the explicit expression of $\delta\Omega^{MS}$:

$$E^{MS} = \frac{<\delta\theta^2>}{||\vec{v}||^2}\left(<cos^2\varphi> Z <p^{in\ t}(p^{in})> Z + <sin^2\varphi> V <p^{in\ t}(p^{in})> V\right)$$

We get:

$$<p^{in\ t}(p^{in})> = <(\bar{p}^{\,in}+\delta p^{in})({}^t\bar{p}^{\,in} + {}^t\delta p^{in})> = \bar{p}^{\,in\ t}(\bar{p}^{\,in}) + <\delta p^{in\ t}(\delta p^{in})>$$

and:

$$E^{MS} = \frac{\sigma_\theta^2}{2||\vec{v}||^2}\left(Z\bar{p}^{\,in\ t}(\bar{p}^{\,in})Z + V\bar{p}^{\,in\ t}(\bar{p}^{\,in})V + ZE^{PP}Z + VE^{PP}V\right)$$

where σ_θ is given by the wellknown formula $\sigma_\theta = \frac{0.014}{p^{in}}\sqrt{\frac{dx}{X_o}}$ with P^{in} in GeV.

Considering the energy loss term and assuming that the energy loss fluctuations are gaussian, we get that:

$$<\left(\frac{\delta e_{loss}}{||\vec{p}^{in}||}\right)^2 p^{in\ t}(p^{in})> = <\left(\frac{\delta e_{loss}}{||\vec{p}^{in}||}\right)^2> \left(\bar{p}^{\,in\ t}(\bar{p}^{\,in}) + E^{PP}\right) = \frac{1}{||\vec{p}^{in}||^2}\frac{dx}{L}\sigma^2_{E_{loss}}\left(\bar{p}^{\,in\ t}(\bar{p}^{\,in}) + E^{PP}\right).$$

Finally, the propagation for the momentum part of the error matrix E^{PP} reads:

$$\begin{aligned}
E^{PPout} &= E^{PPin} - E^{PPin}\delta\Omega^B + \delta\Omega^B E^{PPin} - \delta\Omega^B E^{PPin}\delta\Omega^B \\
&\quad - \frac{\sigma_\theta^2}{2||\vec{v}||^2}\left(Z\bar{p}^{\,in\ t}(\bar{p}^{\,in})Z + V\bar{p}^{\,in\ t}(\bar{p}^{\,in})V + ZE^{PP}Z + VE^{PP}V\right) \\
&\quad + \frac{1}{||\vec{p}^{in}||^2}\frac{dx}{L}\sigma^2_{E_{loss}}\left(\bar{p}^{\,in\ t}(\bar{p}^{\,in}) + E^{PP}\right).
\end{aligned}$$

2.2 Propagation of E^{XX}:

For the position variables, we get:

$$\begin{aligned}
E^{XX\,out} &= <\delta x^{out}\, {}^t(\delta x)^{out}> \\
\text{with } \delta x^{out} &= \delta x^{in} + dx\left(\delta u^{in} + \frac{\delta\Omega^B}{2}\delta u^{in} + \frac{\delta\Omega^{MS}}{2}u^{in}\right) \\
\text{and } {}^t\delta x^{out} &= {}^t(\delta x^{in}) + dx\left({}^t\delta u^{in} - {}^t\delta u^{in}\frac{\delta\Omega^B}{2} - {}^t u^{in}\frac{\delta\Omega^{MS}}{2}\right)
\end{aligned}$$

where $u^{in} = \frac{p^{in}}{||\vec{p}^{in}||}$. Hence, taking into account that $<\delta\Omega^{MS}>= 0$, at first order in $\delta\Omega^B$:

$$\begin{aligned}
E^{XX\,out} &= <\delta x^{in}\, {}^t(\delta x^{in})> + dx <\delta x^{in}\, {}^t(\delta u^{in})> \left(1 - \frac{\delta\Omega^B}{2}\right) \\
&+ dx\left(1 + \frac{\delta\Omega^B}{2}\right) <\delta u^{in}\, {}^t(\delta x^{in})> \\
&+ dx^2\left(<\delta u^{in}\, {}^t(\delta u^{in})> + \frac{\delta\Omega^B}{2} <\delta u^{in}\, {}^t(\delta u^{in})> - <\delta u^{in}\, {}^t(\delta u^{in})> \frac{\delta\Omega^B}{2}\right) \\
&- \frac{1}{4}\left(\frac{dx}{||\vec{p}^{in}||}\right)^2 \left(<\delta\Omega^{MS} p^{in}\, {}^t(p^{in})\delta\Omega^{MS}>\right).
\end{aligned}$$

And with $\delta(\frac{\vec{p}}{p}) = \frac{\delta\vec{p}}{p} - \frac{\vec{p}.\delta\vec{p}}{p^3}\vec{p}$:

$$\begin{aligned}
\delta u^{in} &= \frac{\delta p^{in}}{||\vec{p}^{in}||} - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^3}\delta p^{in} \\
{}^t(\delta u^{in}) &= \frac{{}^t(\delta p^{in})}{||\vec{p}^{in}||} - {}^t(\delta p^{in})\frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^3}
\end{aligned}$$

We get:

$$\begin{aligned}
<\delta u^{in}\, {}^t(\delta u^{in})> &= \frac{<\delta p^{in}\, {}^t(\delta p^{in})>}{||\vec{p}^{in}||^2} + \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^3} <\delta p^{in}\, {}^t(\delta p^{in})> \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^3} \\
&\quad - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^4} <\delta p^{in}\, {}^t(\delta p^{in})> - <\delta p^{in}\, {}^t(\delta p^{in})> \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^4} \\
&= \left(1 - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PPin}}{||\vec{p}^{in}||^2}\left(1 - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^2}\right)
\end{aligned}$$

$$\begin{aligned}
<\delta x^{in}\, {}^t(\delta u^{in})> &= <\delta x^{in}\frac{{}^t(\delta p^{in})}{||\vec{p}^{in}||}> - <\delta x^{in}\frac{{}^t(\delta p^{in})p^{in}}{||\vec{p}^{in}||^3}\, {}^t p^{in}> \\
&= \frac{E^{XPin}}{||\vec{p}^{in}||}\left(1 - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^2}\right) \\
<\delta u^{in}\, {}^t(\delta x^{in})> &= \left(1 - \frac{p^{in}\, {}^t p^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PXin}}{||\vec{p}^{in}||}
\end{aligned}$$

Finally:

$$
\begin{aligned}
E^{XXout} &= E^{XXin} + \frac{dx}{||\vec{p}^{in}||}\left(E^{XPin}(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}).(1-\frac{\delta\Omega^B}{2}) + (1+\frac{\delta\Omega^B}{2}).(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2})E^{PXin}\right) \\
&+\frac{dx^2}{2}\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PPin}}{||\vec{p}^{in}||^2}\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right).\left(1-\delta\Omega^B\right) \\
&+\frac{dx^2}{2}\left(1+\delta\Omega^B\right).\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PPin}}{||\vec{p}^{in}||^2}\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right) \\
&-\frac{1}{4}\left(\frac{dx}{||\vec{p}^{in}||}\right)^2 E^{MS}.
\end{aligned}
$$

2.3 Propagation of E^{XP}:

The last submatrix to propagate is the part combining position and momentum variables:

$$
\begin{aligned}
E^{XPout} &= <\delta x^{out}\,{}^t(\delta p^{out})> \\
\text{with } \delta x^{out} &= \delta x^{in} + dx\left(\delta u^{in} + \frac{\delta\Omega^B}{2}\delta u^{in} + \frac{\delta\Omega^{MS}}{2}u^{in}\right) \\
\text{and } {}^t(\delta p^{out}) &= {}^t(\delta p^{in}) - {}^t(\delta p^{in})\delta\Omega^B - {}^t(p^{in})\delta\Omega^{MS} + \frac{\delta e_{loss}}{||\vec{p}^{in}||}{}^t(p^{in}).
\end{aligned}
$$

In the development of E^{XPout}, there is no energy loss term, since δe_{loss} is only present at first power and we know that $<\delta e_{loss}>=0$. It reads:

$$
\begin{aligned}
E^{XPout} &= <\delta x^{in}\,{}^t(\delta p^{in})> - <\delta x^{in}\,{}^t(\delta p^{in})>\delta\Omega^B - <\delta x^{in}\,{}^t(p^{in})\delta\Omega^{MS}> \\
&+dx\left(<\delta u^{in}\,{}^t(\delta p^{in})> - <\delta u^{in}\,{}^t(\delta p^{in})>\delta\Omega^B - <\delta u^{in}\,{}^t(p^{in})\delta\Omega^{MS}>\right) \\
&+dx\left(\frac{\delta\Omega^B}{2}<\delta u^{in}\,{}^t(\delta p^{in})>(1-\delta\Omega^B) - \frac{\delta\Omega^B}{2}<\delta u^{in}\,{}^t(p^{in})\delta\Omega^{MS}>\right) \\
&-\frac{dx}{2||\vec{p}^{in}||}\left(<\delta\Omega^{MS}p^{in}\,{}^t(p^{in})\delta\Omega^{MS}>\right)
\end{aligned}
$$

with:

$$
<\delta u^{in}\,{}^t(\delta p^{in})> = \left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PPin}}{||\vec{p}^{in}||}
$$

Hence, with $<\delta\Omega^{MS}>=0$ and at first order in $\delta\Omega^B$:

$$
\begin{aligned}
E^{XPout} &= E^{XPin}(1-\delta\Omega^B) + dx\left(1+\frac{\delta\Omega^B}{2}\right)\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{in}||^2}\right)\frac{E^{PPin}}{||\vec{p}^{in}||}\left(1-\delta\Omega^B\right) \\
&-\frac{dx}{2||\vec{p}^{in}||}E^{MS}.
\end{aligned}
$$

Knowing that $E^{PXout} = {}^tE^{XPout}$, we get:

$$\begin{aligned} E^{PXout} &= (1+\delta\Omega^B)E^{PXin} + dx\left(1+\delta\Omega^B\right)\frac{E^{PPin}}{||\vec{p}^{\,in}||}\left(1-\frac{p^{in}\,{}^tp^{in}}{||\vec{p}^{\,in}||^2}\right)\left(1-\frac{\delta\Omega^B}{2}\right) \\ &\quad -\frac{dx}{2||\vec{p}^{\,in}||}E^{MS}. \end{aligned}$$

3 Combination of Muon Error Matrix with Central Tracker Error Matrix

In this section, we explain how we combine local muon error matrix with central tracker error matrix. We perform the combination either at the distance of closest approach to the beam axis (DCA) or at the Preshower radius.

First we have to propagate the error matrix in (px, py, px, x, y, z) coordinates through the toroid and the calorimeter. Then we transform the propagated muon error matrix from the coordinates (px, py, px, x, y, z) to the central tracker variables: $(r, z, \phi, tan\lambda, \frac{q}{P_t})$ for a combination at the DCA point or $(\phi_{pos}, z, \alpha, tan\lambda, \frac{q}{P_t})$ for a combinaison at a given radius (see *D0 Note 3618*). Finally we can combine the two tracks and the two error matrices to obtain the final track parameters.

3.1 Coordinate Change: from Propagation Coordinates to DCA Central Tracker Coordinates

During the propagation (see Section 2), we have computed the error matrix in the 6 variables: (px, py, px, x, y, z). The comparison and/or combination with an other measurement (e.g. from central tracker) is performed at a location given by an additional constraint. At the DCA, it is: $V = \vec{r}.\vec{p} = x.Px + y.P_y = 0$. At this location, the error matrix should be given in the 5 remaining independent variables (V fixed). For the combination at the DCA, we have first to make the coordinate change (px, py, px, x, y, z) to $(r, z, \phi, tan\lambda, \frac{q}{P_t}, V)$ where the 5 first variables are the central tracker variables at this point. Here r is the signed impact parameter, ϕ and λ are related to the track direction as defined in the *D0 Note 3618*. The definition of the DCA point implies that the track point position in the plane (x, y) is orthogonal to the track direction. So we have:

$$\phi = atan2(P_y, P_x) = -atan2(x, y) \text{ with } -\pi \leq \phi \leq \pi.$$

We can write with a signed r:

$$\begin{aligned} x &= r.sin\phi \\ y &= -r.cos\phi \end{aligned}$$

where the sign of r is given by: $sign = sign(\frac{-y}{cos\phi}) = sign(\frac{x}{sin\phi}) = sign(xP_y)$. With those conventions, we calculate the central tracker coordinates with respect to the previous ones:

$$
\begin{aligned}
r &= sign.\sqrt{x^2 + y^2} \\
z &= z \\
\phi &= atan2(P_y, P_x) \\
tan\lambda &= \frac{P_z}{\sqrt{P_x^2 + P_y^2}} \\
\frac{q}{P_t} &= \frac{q}{\sqrt{P_x^2 + P_y^2}} \\
V &= xP_x + yP_y
\end{aligned}
$$

The jacobian matrix for the coordinate change is:

$$
J = \frac{\partial(r, z, \phi, tan\lambda, \frac{q}{P_t}, V)}{\partial(px, py, px, x, y, z)} = \begin{pmatrix} 0 & 0 & 0 & \frac{x}{r} & \frac{y}{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-P_y}{P_t^2} & \frac{P_x}{P_t^2} & 0 & 0 & 0 & 0 \\ \frac{-P_x P_z}{P_t^3} & \frac{-P_y P_z}{Pt^3} & \frac{1}{P_t} & 0 & 0 & 0 \\ \frac{-qP_x}{P_t^3} & \frac{-qP_y}{P_t^3} & 0 & 0 & 0 & 0 \\ x & y & 0 & P_x & P_y & 0 \end{pmatrix}
$$

So the new error matrix E_{r6} in $(r, z, \phi, tan\lambda, \frac{q}{P_t}, V)$ is related to the old one E_x in (px, py, px, x, y, z) by:

$$
E_{r6} = J.E_x.{}^tJ
$$

Finally we obtain the reduced (5×5) error matrix E_r by forgetting the last line and the last colomn of E_{r6}.

3.2 Coordinate Change: from Propagation Coordinates to Fixed Radius Central Tracker Coordinates

Here we want to compare the error matrix in the 6 variables (px, py, px, x, y, z) coming from the muon error matrix propagation with the central tracker error matrix at a fixed radius (Preshower radius). The constraint at this location is: $R = \sqrt{x^2 + y^2} =$ fixed value. The error matrix should be given in the 5 remaining independent variables (R fixed). So first, we have to make the coordinate change (px, py, px, x, y, z) to $(r, \phi_{pos}, z, \alpha, tan\lambda, \frac{q}{P_t})$ where the last 5 variables are the central tracker variables. Here $\alpha = \phi_{dir} - \phi_{pos}$ as defined in the *D0 Note 3618*. So we calculate the central tracker coordinates with respect to the previous ones:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\phi_{pos} &= atan2(y, x) \\
z &= z \\
\alpha &= atan2(P_y, P_x) - atan2(y, x)
\end{aligned}
$$

$$
\begin{aligned}
tan\lambda &= \frac{P_z}{\sqrt{P_x^2+P_y^2}} \\
\frac{q}{P_t} &= \frac{q}{\sqrt{P_x^2+P_y^2}}
\end{aligned}
$$

The jacobian matrix for the coordinate change is:

$$
J = \frac{\partial(r,\phi_{pos},z,\alpha,tan\lambda,\frac{q}{P_t})}{\partial(px,py,px,x,y,z)} = \begin{pmatrix}
0 & 0 & 0 & \frac{x}{r} & \frac{y}{r} & 0 \\
0 & 0 & 0 & \frac{-y}{r} & \frac{x}{r} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{-P_y}{P_t^2} & \frac{P_x}{P_t^2} & 0 & \frac{y}{r} & \frac{-x}{r} & 0 \\
\frac{-P_xP_z}{P_t^3} & \frac{-P_yP_z}{Pt^3} & \frac{1}{P_t} & 0 & 0 & 0 \\
\frac{-qP_x}{P_t^3} & \frac{-qP_y}{P_t^3} & 0 & 0 & 0 & 0
\end{pmatrix}
$$

So the new error matrix E_{r6} in $(r,\phi_{pos},z,\alpha,tan\lambda,\frac{q}{P_t})$ is related to the old one E_x in (x,y,z,px,py,px) by:

$$
E_{r6} = J.E_x.{}^tJ
$$

Finally we obtain the reduced (5×5) error matrix E_r by forgetting the first line and the first colomn of E_{r6}.

3.3 Error Matrices Combination:

To compute the final track parameters P_f and the final error matrix E_f, we have to combine the propagated muon track parameters P_r and the propagated muon error matrix E_r computed in the previous section with the central tracker parameters P_c and the central tracker error matrix E_c. Those combination is performed using the following statistical formulae:

$$
\begin{aligned}
P_f &= \left(E_r^{-1}+E_c^{-1}\right)^{-1}.\left(E_r^{-1}.P_r+E_c^{-1}.P_c\right) \\
E_f &= \left(E_r^{-1}+E_c^{-1}\right)^{-1}
\end{aligned}
$$

We also can define a χ^2 conveying the quality of the previous combination:

$$
\chi^2 = {}^t(P_r-P_c).(E_r+E_c)^{-1}.(P_r-P_c)
$$

Appendix

A cross product is transformed into a matrix product (antisymmetric matrix times column vector) in the following way:

$$
\vec{P} \longrightarrow \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}
$$

$$\vec{B} \times \vec{P} = \left(\begin{array}{c} B_y P_z - B_z P_y \\ B_z P_x - B_x P_z \\ B_x P_y - B_y P_x \end{array} \right) \longrightarrow B.P = \left(\begin{array}{ccc} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{array} \right) \left(\begin{array}{c} P_x \\ P_y \\ P_z \end{array} \right)$$